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The Asymptotics of a Continuous Analogue of Orthogonal Polynomials

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Szegö polynomials are associated with weight functions on the unit circle. M. G. Krein introduced a continuous analogue of these, a family of entire functions of exponential type associated with a weight function on the real line. An investigation of the asymptotics of the resolvent kernel of $\sin(x-y)/\pi(x-y)$ on $[0, s]$ leads to questions of the asymptotics of the Krein functions associated with the characteristic function of the complement of the interval $[-1, 1]$. Such asymptotics are determined here, and this leads to answers to certain questions involving the above-mentioned kernel, questions arising in the theory of random matrices.

1. INTRODUCTION

The continuous analogue of the (Szegö) orthogonal polynomials was introduced by M. G. Krein [4]. These are families of entire functions of exponential type which were shown by him to have some, but not all, properties analogous to those enjoyed by the orthogonal polynomials.

If ρ is a nonnegative weight function defined on the unit circle then the sequence of coefficients of the suitably normalized associated Szegö polynomials $P_n(z)$ is given by the vector $T_n(\rho)^{-1} e_n$, where $T_n(\rho)$ is the $(n+1)\times(n+1)$ Toeplitz matrix associated with ρ and $e_n = (0, ..., 0, 1)$ is the last vector of a standard basis. Analogously, the inverse Fourier transform of the Krein function $P(s, \xi)$ is the solution of the integral equation

$$g(x) - \int_0^s k(x-y)\, g(y)\, dy = k(x-s) \tag{1.1}$$

on $L_2(0, s)$.

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The particular kernel

$$k(x)=\frac{\sin x}{\pi x} \tag{1.2}$$

arises in many areas of mathematics and mathematical physics. In particular, the Fredholm determinant

$$\det(I-K_s), \tag{1.3}$$

where K_s denotes the operator with kernel $k(x-y)$ on $L_2(0,s)$, and the related quantity

$$-\frac{d}{d\lambda}\det(I-\lambda K_s)|_{\lambda=1}=\operatorname{tr} K_s(I-K_s)^{-1}\cdot\det(I-K_s), \tag{1.4}$$

arise in the theory of random matrices [5, Chap. 5]. The first is the probability, in a certain model, that a hermitian matrix has no eigenvalues in a given interval of length s, and the second is the probability that there is exactly one.

The main results of this paper are proofs of the two first-order asymptotic formulas

$$\operatorname{tr} K_s(I-K_s)^{-1}\sim\frac{e^s}{2\sqrt{2\pi s}} \tag{1.5}$$

$$\log\det(I-K_s)\sim -\frac{s^2}{8}. \tag{1.6}$$

(These mean, as usual, that the ratios of the two sides tend to 1 as $s\to\infty$.)

A complete expansion, of which (1.6) was the first term, was obtained by Dyson [2]. The derivation, which used methods of inverse scattering, was quite formal. Interestingly, the constant term in the expansion could not be obtained this way but was derived by a (formal) scaling argument from a result on the asymptotics of a certain Toeplitz determinant [6]. In [1] it was shown how to derive the Toeplitz analogue of (1.5) through the asymptotics of the polynomials orthogonal on a circular arc, and (1.5) itself results from the same formal scaling argument. It was also pointed out there how these first-order results could be used to determine all coefficients (except for the constant term alluded to above) in presumed asymptotic expansions.

Thus, actual proofs of (1.5) and (1.6) are presented here for the first time. What we need are good asymptotics for the kernel of $K_s(I-K_s)^{-1}$, and a theorem of Gohberg and Semencul [3, Th. III.8.1] shows the beginning of the path to this goal. It gives the formula for the kernel of the operator

$K_s(I-K_s)^{-1}$, for an arbitrary convolution kernel $k(x-y)$, in terms of the solution to a modification of Eq. (1.1),

$$\gamma_+(x)-\int_0^s k(x-y)\,\gamma_+(y)\,dy=k(x) \tag{1.7}$$

and the solution of

$$\gamma_-(x)-\int_0^s k(y-x)\,\gamma_-(y)\,dy=k(-x). \tag{1.8}$$

The formula for $R_s(x, y)$, the kernel of $K_s(I-K_s)^{-1}$, is

$$R_s(x, y)=\gamma_{\pm}(|x-y|)+\int_0^{\min(x,y)} \gamma_+(x-z)\,\gamma_-(y-z)\,dz$$
$$-\int_s^{s+\min(x,y)} \gamma_+(z-x)\,\gamma_-(z-y)\,dz, \tag{1.9}$$

where in the first term on the right $\pm=\operatorname{sgn}(x-y)$. In our case k is even and so $\gamma_+(x)=\gamma_-(x)=g(s-x)$, where g is the solution to (1.1). So what we need in order to prove (1.5) are the asymptotics of a certain family of Krein functions.

There is a well-developed asymptotic theory for Szegö polynomials and it turns out, not surprisingly, that some of the ideas can be used for Krein functions as well. For the Szegö polynomials the classical method uses the fact that they solve an extremal problem. One first finds a good approximation to the extremal quantity and then produces a polynomial that gives this good approximation. This must then be close to the Szegö polynomial. There seems to be no analogous extremal quantity for the Krein functions but we found that we could use instead a certain kind of "reproducing" property (see Lemma 1 below) as a substitute. Another awkwardness is that in the case of (1.2) the Fourier transform of $\delta(t)-k(t)$ (note that $I-K_s$ is convolution by this) is not supported on the full real line, $\mathbb{R}$. In fact, it is the characteristic function of the complement in $\mathbb{R}$ of the interval $[-1, 1]$. Thus we have the continuous analogue of the polynomials orthogonal on a circular arc, rather than the full circle. This is an added complication, but the difficulties are not serious. We shall not work with the Krein function $P(s, \xi)$ directly, but rather the Fourier transform $\varphi(\xi)$ of the solution to (1.7). The relation between these functions is

$$P(s, \xi)=e^{is\xi}[1+\varphi(-\xi)]$$

and the asymptotics of $\varphi(\xi)$ are given in Lemmas 5 and 6(b).

Since, as is well known, the logarithmic derivative of (1.3) with respect to s equals $-R_s(s, s)$, it should come as no surprise that we can derive (1.6) also by these methods.

To end this introduction we acknowledge with pleasure our gratitude to Estelle Basor and Craig Tracy, who brought our attention to the problem of the asymptotics of (1.5).

2. Proof of (1.5)

We denote by γ the common solution of Eq. (1.7) and (1.8). In terms of this solution formula (1.9) for the resolvent kernel becomes

$$R_s(x, y) = \gamma(|x-y|) + \int_0^{\min(x, y)} \gamma(x-z)\, \gamma(y-z)\, dz - \int_s^{s+\min(x, y)} \gamma(z-x)\, \gamma(z-y)\, dz, \tag{2.1}$$

and it follows that

$$\operatorname{tr} K_s(I-K_s)^{-1} = s\gamma(0) + \int_0^s (s-2x)\, \gamma(x)^2\, dx. \tag{2.2}$$

The main contribution will come from the integral. If we set

$$\varphi(\xi) = \hat{\gamma}(\xi) = \int_0^s e^{i\xi x} \gamma(x)\, dx$$

then the integral in question equals

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\varphi(\xi)}\, [s\varphi(\xi) + 2i\varphi'(\xi)]\, d\xi. \tag{2.3}$$

We shall write

$$E = (-\infty, -1) \cup (1, \infty).$$

Lemma 1. *Suppose f is a smooth function defined on $[0, s]$, extended to be 0 outside the interval. Set $\psi = \hat{f}$. Then*

$$\int_E [1 + \varphi(\xi)]\, \overline{\psi(\xi)}\, d\xi = \int_{-\infty}^{\infty} \overline{\psi(\xi)}\, d\xi,$$

both integrals being principal value integrals at ∞.

Proof. We have, since $\hat{k}=\chi_{[-1,1]}$,

$$\int_E \overline{\psi(\xi)}\,d\xi=\int_{-\infty}^{\infty}\overline{\psi(\xi)}\,d\xi-\int_{-\infty}^{\infty}\chi_{[-1,1]}(\xi)\,\overline{\psi(\xi)}\,d\xi$$

$$=\int_{-\infty}^{\infty}\overline{\psi(\xi)}\,d\xi-2\pi\int_0^s k(x)\,\overline{f(x)}\,dx,$$

and similarly,

$$\int_E \varphi(\xi)\,\overline{\psi(\xi)}\,d\xi=2\pi\int_0^s \gamma(x)\,\overline{f(x)}\,dx-2\pi\int_0^s (K_s\gamma)(x)\,\overline{f(x)}\,dx.$$

Since $K_s\gamma=\gamma-k$ the assertion follows. ∎

We now introduce a function which has the same reproducing property as φ. Not being entire, it will certainly not equal φ, but it will be a good approximation to φ on E^c, the complement of E in the complex plane. This function is

$$h(\xi)=e^{(1/2)\,is(\xi-\sqrt{\xi^2-1})}\sqrt{\frac{\xi+\sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}}}.$$

We define $\sqrt{\xi^2-1}$ as that branch, analytic in E^c, which is asymptotically equal to ξ as $\xi\to\infty$ in the upper half plane. For $\xi\in E$ we write

$$h_{\pm}(\xi)=\lim_{\varepsilon\to 0+} h(\xi\pm i\varepsilon).$$

LEMMA 2. *With ψ as in Lemma 1 we have*

$$\int_E [h_+(\xi)+h_-(\xi)]\,\overline{\psi(\xi)}\,d\xi=\int_{-\infty}^{\infty}\overline{\psi(\xi)}\,d\xi.$$

Proof. The variable change

$$t=\xi-\sqrt{\xi^2-1} \tag{2.4}$$

takes E^c to the lower half-plane. Since

$$t^{-1}=\xi+\sqrt{\xi^2-1},\qquad \xi=\tfrac{1}{2}(t^{-1}+t),\qquad \sqrt{\xi^2-1}=\tfrac{1}{2}(t^{-1}+t)$$

the integral in question becomes

$$\int_{-\infty}^{\infty} e^{(1/2)\,ist}\frac{1}{\sqrt{1-t^2}}\,\overline{\psi(\tfrac{1}{2}(t^{-1}+t))}\,\tfrac{1}{2}\,|1-t^{-2}|\,dt,$$

where $\sqrt{1-t^2}$ is that branch which equals 1 when $t=0$ and extends analytically into the lower half-plane. We can write

$$|1-t^{-2}|=t^{-2}\sqrt{1-t^2}\sqrt{1-t^2},$$

where both square roots equal 1 when $t=0$, but one extends analytically into the upper half-plane, the other into the lower. Hence the integral itself can be written

$$\tfrac{1}{2}\int_{-\infty}^{\infty} e^{(1/2)ist}\,\overline{\psi(\tfrac{1}{2}(t^{-1}+t))}\,\sqrt{1-t^2}\,t^{-2}\,dt, \tag{2.5}$$

the square root now being the one that extends analytically into the upper half-plane. But for real ξ

$$\overline{\psi(\xi)}=\int_0^s e^{-i\xi x}\,\overline{f(x)}\,dx$$

and this extends to an entire function of ξ which is $O(e^{s\,\mathrm{Im}\,\xi}/|\xi|)$ as $\xi\to\infty$ in the upper half-plane and is

$$\frac{\overline{f(0)}}{i\xi}+O(|\xi|^{-2})$$

as $\xi\to\infty$ in the lower half-plane. It follows that the integrand in (2.5) extends analytically into the upper half-plane, is $O(|t|^{-2})$ as $t\to\infty$, and is

$$\frac{2t}{i}\,\overline{f(0)}+O(|t|^2)$$

as $t\to 0$. The assertion follows from these facts, if we use the obvious contour with two semi-circles and the relation

$$\pi f(0)=\int_{-\infty}^{\infty}\psi(\xi)\,d\xi. \quad \blacksquare$$

LEMMA 3. *For every $f\in L_2(0,s)$, with $\psi=\hat{f}$, we have*

$$\int_E [1+\varphi(\xi)-h_+(\xi)-h_-(\xi)]\,\overline{\psi(\xi)}\,d\xi=0.$$

Proof. The assertion is immediate from Lemmas 1 and 2, the fact that the functions φ, $1-h_+$, and h_- all belong to $L_2(-\infty,\infty)$, and the density in $L_2(0,s)$ of the smooth functions. $\blacksquare$

We want to deduce from Lemma 3 that $1+\varphi$ is close to $h_+ + h_-$ on E and to do this we have to replace $h_+(\xi)+h_-(\xi)$ by an entire function with the same general behavior as φ and which is close to $h_+ + h_-$ on E. Such a function will be $1+q(\xi)$, where

$$q(\xi)=\frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty}\frac{h(\eta)}{\eta-\xi}\,d\eta,$$

where we take $a<\min(0, \operatorname{Im}\xi)$. Clearly q is entire and it is easily seen that

$$\begin{aligned} q(\xi)&=h(\xi)+O(|\xi|)^{-1} && \text{as } \xi\to\infty, \operatorname{Im}\xi\leqslant 0\\ q(\xi)&=O(|\xi|)^{-1} && \text{as } \xi\to\infty, \operatorname{Im}\xi\geqslant 0. \end{aligned} \tag{2.6}$$

(For the first we move the path of integration to above ξ.)

LEMMA 4. *We have, as* $s\to\infty$,

(a) $\int_E |1+q(\xi)-h_+(\xi)-h_-(\xi)|^2\,d\xi=O(s^{-1})$,

(b) $q(\xi)=h(\xi)+O(1)$ *uniformly on compact subsets of* E^c.

Proof. Using the same variable change (2.4) as before (with η replacing ξ) we obtain

$$q(\xi)=\frac{1}{2\pi i}\int e^{(1/2)ist}\frac{1}{\sqrt{1-t^2}}(1-t^{-2})\frac{dt}{t^{-1}+t-2\xi}, \tag{2.7}$$

where the path of integration can be taken to be a horizontal line, traversed from left to right, well below the real axis. The integrand is single-valued in the plane cut from -1 to $-1+i\infty$ and 1 to $1+i\infty$. It has a pole at $t=0$ with residue -1.

Consider part (a) first, when $\xi\in E$. The integrand has poles also at $\xi\pm\sqrt{\xi^2-1}$ on the real line with residues $h_\pm(\xi)$. Let C_+ denote the cut from 1 to $1+i\infty$ traversed "counterclockwise," i.e., once in each direction, using the two limiting values of the integrand in the usual way. Similarly C_- denotes the cut from -1 to $-1+i\infty$ traversed counterclockwise. Since the integrand is $O(|t|^{-2})$ as $t\to\infty$ in the upper half-plane we see easily that

$$1+g(\xi)=h_+(\xi)+h_-(\xi)+\frac{1}{2\pi i}\left(\int_{C_-}+\int_{C_+}\right).$$

To conclude the proof of (a) we must also show that

$$\int_E\left|\int_{C_\pm}\cdots dt\right|^2 d\xi=O(s^{-1}).$$

We have

$$\int_{C_\pm} = 2\int_{\pm 1}^{\pm 1 + i\infty} e^{(1/2)ist} \frac{1}{\sqrt{1-t^2}} \left(1 - \frac{1}{t^2}\right) \frac{dt}{t^{-1} + t - 2\xi}$$

with an appropriate branch of the square root. If we write $t = \pm 1 + iu$ the part of the integral corresponding to $u > \delta$ (for any $\delta > 0$) is a function which is exponentially small in s times $|\xi|^{-1}$. The integral over $u < \delta$ is bounded by a constant times

$$\int_0^\delta e^{-(1/2)su} \frac{\sqrt{u}}{u^2 + \xi^2 - 1}\, du$$

and it is an easy exercise that the L_2 norm of this over $\xi \in E$ in $O(s^{-1/2})$.

For part (b), when $\xi \in E^c$, the poles other than the one at $t = 0$ are at $\xi - \sqrt{\xi^2 - 1}$ in the lower half-plane and $\xi + \sqrt{\xi^2 - 1}$ in the upper. The former contributes the residue $h(\xi)$ and the other a residue which is exponentially small as $s \to \infty$. This establishes (b), with $O(1)$ equal to 1 plus an exponentially small term. ∎

Lemma 5. $\int_E |1 + \varphi(\xi) - h_+(\xi) - h_-(\xi)|^2\, d\xi = O(s^{-1})$.

Proof. Write the integrand as

$$[1 + \varphi(\xi) - h_+(\xi) - h_-(\xi)]\, \overline{[1 + q(\xi) - h_+(\xi) - h_-(\xi)]} + [1 + \varphi(\xi) - h_+(\xi) - h_-(\xi)]\, \overline{[\varphi(\xi) - q(\xi)]}.$$

It follows from (2.6), the asymptotic behavior of h, and the Paley–Wiener theorem that q is the Fourier transform of a function in $L_2(0, s)$, as is φ. Hence by Lemma 3 the integral over E of the second term above is 0. By Lemma 4(a) the square of the integral of the first term has absolute value at most a constant times

$$s^{-1} \int_E |1 + \varphi(\xi) - h_+(\xi) - h_-(\xi)|^2\, d\xi.$$

The integral here is exactly what we started with, so the result follows. ∎

The next lemma will tell us that the major contribution to the integral in (2.3) will come from the interior of the interval $[-1, 1]$, and that φ is very close to h there.

Lemma 6. (a) *For any* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that*

$$\int_{|\xi| > 1-\delta} [|\varphi(\xi)|^2 + |\varphi'(\xi)|^2]\, d\xi = O(e^{\varepsilon s}) \qquad \text{as} \quad s \to \infty.$$

(b) *$\varphi(\xi)=h(\xi)(1+O(s^{-1/2}))$ as $s\to\infty$ uniformly on compact subsets of E^c.*

Proof. We begin with (a). It is easy to check that

$$\int_E |1-h_+(\xi)|^2\,d\xi=O(s),\qquad \int_E |h_-(\xi)|^2\,d\xi=O(1) \tag{2.8}$$

and so from Lemma 5

$$\int_E |\varphi(\xi)|^2\,d\xi=O(s). \tag{2.9}$$

The function

$$\theta(\xi)=\varphi(\xi)\,e^{-(1/2)\,is(\xi-\sqrt{\xi^2-1})} \tag{2.10}$$

is analytic in E^c and $O(|\xi|^{-1})$ for large ξ, and so for $\xi\notin E$

$$\theta(\xi)=\frac{1}{2\pi i}\oint_E \frac{\theta(\eta)}{\eta-\xi}\,d\xi. \tag{2.11}$$

Here $\oint$ indicates that E is traversed twice, once in each direction, using the appropriate limiting values of θ in the integrand. Let Γ denote the positively oriented contour (actually, the union of two contours) consisting of the points whose distance to E is 2δ. Thus Γ has two semi-circular parts and four horizontal parts. We have, of course, from (2.9) that

$$\int_E |\theta(\eta)|^2\,d\theta=O(s).$$

From this and the integral representation (2.11) we deduce that

$$\int_\Gamma |\theta(\xi)|^2\,|d\xi|=O(s).$$

(For the integrals over the semi-circular parts of Γ this is immediate; for the horizontal parts we use the uniform boundedness of the operators

$$f(\xi)\to\int_{-\infty}^{\infty}\frac{f(n)\,d\eta}{\eta-\xi\pm 2i\delta}$$

on $L_2(-\infty,\infty)$.) And from this it follows that

$$\int_\Gamma |\varphi(\xi)|^2\,|d\xi|=O(se^{(1/2)\,\varepsilon s}),$$

where

$$\varepsilon = \sup_{\Gamma} |\mathrm{Im}(\xi - \sqrt{\xi^2 - 1})|.$$

Clearly $\varepsilon \to 0$ as $\delta \to 0$. Finally we use the representations

$$\varphi(\xi) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\eta)}{\eta - \xi}\, d\eta, \qquad \varphi'(\xi) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\eta)}{(\eta - \xi)^2}\, d\eta$$

for $\xi \in (-\infty, -1+\delta) \cup (1-\delta, \infty)$, the last estimate, and the argument just preceding it, to deduce the assertion of part (a).

For the proof of (b) we start with

$$\int_E |\varphi(\xi) - q(\xi)|^2\, d\xi = O(s^{-1}),$$

which follows from Lemmas 4(a) and 5. This is equivalent to

$$\oint_E |[\varphi(\xi) - q(\xi)]\, e^{(1/2)\, is(\xi - \sqrt{\xi^2 - 1})}|^2\, |d\xi| = O(s^{-1}),$$

from which we deduce (by mapping E^c to the unit disc, for example, and using the fact that the integrand here is bounded in E^c) that

$$[\varphi(\xi) - q(\xi)]\, e^{(1/2)\, is(\xi - \sqrt{\xi^2 - 1})} = O(s^{-1/2})$$

uniformly on compact subsets of E^c. Assertion (b) follows from this, Lemma 4(b), and the fact that $|h(\xi)|/s^{1/2} \to \infty$ uniformly on compact subsets of E^c. ∎

LEMMA 7. *We have*

$$\int_{-\infty}^{\infty} \overline{\varphi(\xi)}\, [s\varphi(\xi) + 2i\varphi'(\xi)]\, d\xi \sim \sqrt{\frac{\pi}{2s}}\, e^s \qquad \textit{as} \quad s \to \infty.$$

Proof. By Lemma 6(a) it suffices to prove this asymptotic result when the domain of integration is replaced by $(-1+\delta, 1-\delta)$ for some small $\delta > 0$. If we define $\theta(\xi)$ by (2.10) as before then Lemma 6(b) tells us that

$$\theta(\xi) = \sqrt{\frac{\xi + \sqrt{\xi^2 - 1}}{2\sqrt{\xi^2 - 1}}}\, (1 + O(s^{-1/2})) \tag{2.12}$$

uniformly on compact subsets of E^c. It follows automatically from this that

$$\frac{\theta'(\xi)}{\theta(\xi)}=\frac{d}{d\xi}\log\sqrt{\frac{\xi+\sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}}}+O(s^{-1/2}) \tag{2.13}$$

uniformly on compact subsets of E^c, and so uniformly on $(-1+\delta, 1-\delta)$. Using the general fact

$$\overline{f(\xi)}\,f'(\xi)=|f(\xi)|^2\frac{d}{d\xi}\log f(\xi)$$

we can write our integrand as

$$e^{s\sqrt{1-\xi^2}}\,|\theta(\xi)|^2\left[s-s\left(1+\frac{i\xi}{\sqrt{1-\xi^2}}\right)+2i\,\frac{\theta'(\xi)}{\theta(\xi)}\right]$$
$$=e^{s\sqrt{1-\xi^2}}\,|\theta(\xi)|^2\left[-\frac{i\xi}{\sqrt{1-\xi^2}}\,s+2i\,\frac{\theta'(\xi)}{\theta(\xi)}\right].$$

(We also used the fact that $\sqrt{\xi^2-1}=i\sqrt{1-\xi^2}$ for $\xi\in(-1,1)$, the latter square root being positive.) Since the expression outside the brackets is even and the first term in brackets is odd its contribution to the integral over $(-1+\delta, 1-\delta)$ is zero. Using (2.9) and (2.10) we see that the integral in question is asymptotically

$$2i\,|\theta(0)|^2\,\frac{\theta'(0)}{\theta(0)}\sqrt{\frac{2\pi}{s}}\,e^s.$$

As is easily seen, $|\theta(0)|=1/\sqrt{2}$ and $\theta'(0)/\theta(0)=1/2i$, and the result follows. ∎

All that remains to finish the proof of (1.5) is to show that the first term on the right side of (2.2) is $o(e^s/s^{3/2})$. In fact we shall show in the next section that it is of order s^2, but the much cruder estimate is all we need now and can be obtained from what we have already done, as follows. From Lemma 6 and the behavior of h it is trivial that

$$\int_{-\infty}^{\infty}|\varphi(\xi)|^2\,d\xi=O(e^s s^{-1/2}),$$

which is of course equivalent to

$$\int_0^s \gamma(x)^2\,dx=O(e^s s^{-1/2}).$$

From this an the defining equation (1.6) for γ it follows that $\gamma(0) = O(e^{s/2}s^{-1/4})$, which is more than good enough.

3. Proof of (1.6)

What we shall prove is that

$$\frac{d}{ds}\log\det(I-K_s) = -\frac{s}{4}+O(1) \tag{3.1}$$

and (1.6) will follow upon integration. As mentioned above we have (a general fact)

$$\frac{d}{ds}\log\det(I-K_s) = -R_s(s,s),$$

and so from (2.1)

$$\frac{d}{ds}\log\det(I-K_s) = -\gamma(0). \tag{3.2}$$

Of course,

$$\gamma(0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(\xi)\,d\xi.$$

If we set $x=0$ in (1.7) and use the fact $\hat{k} = \chi_{[-1,1]}$ we obtain

$$\gamma(0) = \frac{1}{\pi}+\frac{1}{2\pi}\int_{-1}^{1}\varphi(\xi)\,d\xi.$$

Let us estimate the error if we replace φ by q in the integral. We apply Lemma 1, in the equivalent form

$$\int_E \overline{\varphi(\xi)}\,\psi(\xi)\,d\xi = \int_{-1}^{1}\psi(\xi)\,d\xi,$$

to $\psi=\varphi$ and to $\psi=q$ and subtract, obtaining

$$\int_{-1}^{1}[\varphi(\xi)-q(\xi)]\,d\xi = \int_E \overline{\varphi(\xi)}\,[\varphi(\xi)-q(\xi)]\,d\xi. \tag{3.3}$$

Hence

$$\left|\int_{-1}^{1} [\varphi(\xi)-q(\xi)]\,d\xi\right| \leqslant \int_E |\varphi(\xi)|^2\,d\xi \int_E |\varphi(\xi)-q(\xi)|^2\,d\xi.$$

The second integral on the right is $O(s^{-1})$, by Lemmas 4(a) and 5, while the first is $O(s)$, by Lemma 5 and (2.8). Thus

$$\gamma(0)=\frac{1}{\pi}+\frac{1}{2\pi}\int_{-1}^{1} q(\xi)\,d\xi+O(1). \tag{3.4}$$

From the representation (2.7) of $q(\xi)$ it follows that

$$\int_{-1}^{1} q(\xi)\,d\xi=-\frac{1}{2\pi i}\int e^{(1/2)\,ist}\frac{1}{\sqrt{1-t^2}}(1-t^{-2})\log\frac{t-1}{t+1}\,dt, \tag{3.5}$$

where the integral is taken over a horizontal line traversed left to right, below the real axis, and the logarithm is that branch which is analytic in the lower half-plane and which tends to 0 as $t\to\infty$ in the lower half-plane. Recall that the square root is the branch which is analytic in the lower half-plane and which tends to 1 as $t\to 0$. The integrand continues to a single-valued analytic function in the plane cut from -1 to $-1+i\infty$ and from 1 to $1+i\infty$ except for a double pole at $t=0$. As in the proof of Lemma 4(a), we introduce the contours $C_\pm$ which are the cuts traversed twice each. We find that the integrals over $C_\pm$ are $o(1)$ as $s\to\infty$ and so the right side of (3.5) equals $o(1)$ minus the residue of the integrand at $t=0$. This gives, by an easy computation

$$\int_{-1}^{1} q(\xi)\,d\xi=\frac{\pi s}{2}-2+o(1). \tag{3.6}$$

Hence, from (3.4)

$$\gamma(0)=\frac{s}{4}+O(1).$$

In view of (3.2), we have established (3.1), and so (1.6). ∎

The reader may have wondered why we kept the term $1/\pi$ in (3.4) since it was swallowed by the error term $O(1)$. The reason is that one expects $\gamma(0)=s/4+o(1)$ from Dyson's expansion; equivalently, in view of (3.6), the error $O(1)$ in (3.4) is $o(1)$. Our $O(1)$ came from the crude estimation of the right side of (3.3) by means of Schwarz's inequality. A more subtle way of estimating this integral might lead to the correct estimate $o(1)$ with corresponding error term $o(s)$ in (1.6).

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